Finite-Time and Fixed-Time Input-to-State Stability: Explicit and Implicit Approaches

Francisco Lopez-Ramirez, Denis Efimov, Andrey Polyakov, Wilfrid Perruquetti

Finite-Time and Fixed-Time Input-to-State Stability: Explicit and Implicit Approaches

Francisco Lopez-Ramirez[1], Denis Efimov[1,2], Andrey Polyakov[1,2] and Wilfrid Perruquetti[3]

[1] *Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France.*
[2] *ITMO University, 49 Kronverkskiy av., 197101 Saint Petersburg, Russia.*
[3] *École Centrale de Lille, CNRS, UMR 9189 - CRIStAL, Cité Scientifique, 59651 Villeneuve d'Ascq Cedex, France.*

Abstract

The present article gathers the analysis of non-asymptotic convergence rates (finite-time and fixed-time) with the property of input-to-state stability. Theoretical tools to determine this joint property are presented for the case where an explicit ISS Lyapunov function is known, and when it remains in implicit form (e.g. as a solution of an algebraic equation). For the case of finite-time input-to-state stability, necessary and sufficient conditions are given whereas for the fixed-time case only a sufficient condition is obtained. Academic examples and numerical simulations support the obtained results.

1. Introduction

The study of stability and robustness of dynamical systems is at the center of control system theory; for nonlinear systems the most general framework available to perform this study is Lyapunov analysis (LA) [1]. LA uses scalar functions that satisfy some differential properties in order to determine, for a system with no inputs, if a trajectory that starts close to an equilibrium will remain close to this equilibrium (Lyapunov stability). In the case of input systems, it allows to determine if for any initial condition and any bounded input, the system's state will remain bounded (input-to-state stability) [1, 2, 3].

The first works on input-to-state stability (ISS) were developed considering an asymptotic behavior of the solutions [2], this is, that in the absence of inputs or perturbations, the system's solutions tend asymptotically to the equilibrium point. However, nonlinear systems may exhibit *faster* rates of convergence often called non-asymptotic [4, 5, 6, 7]. In the case of finite-time stability, the system's trajectories converge exactly to zero in a finite amount of time [4]; in the case of fixed-time stability exact convergence to the origin occurs in a maximum amount of time that is independent of the system's initial state [8, 9, 10]. Non-asymptotic convergence rates are a major feature in Sliding Mode Control [11, 12, 13, 14] and some further developments in non-asymptotic convergence include a bound on the convergence time that is not only fixed but also arbitrarily selected [15, 16, 17], a better control performance when initial conditions are far away from the origin by separating low and high growing terms [18, 19] and finite-time stable controllers with an enhancement of the domain of attraction for state-constrained systems [20]. Although systems with non-asymptotic rates of convergence may exhibit numerical inconsistencies [21] or lose some of its properties under discretization algorithms [22], recent advances in consistent discretization provide algorithms that overcome these issues [23, 24].

To gather the properties of non-asymptotic stability and robustness with respect to exogenous inputs in a single framework is useful not only for analysis purposes, but also to design controllers and observers.

In [25], a first set of results that gathers ISS with finite-time stability can be found; there, the authors show that the existence of a finite-time ISS Lyapunov function implies the finite-time ISS property, however, this implication only goes in one direction *i.e.* a converse result is not obtained. One of the main contributions of this article is to show that if an assumption about the continuity of the settling-time function is made, a converse result can be obtained.

When studying non-asymptotic stable systems using LA, the differential inequalities that a Lyapunov function satisfies are much more difficult to verify [26], and the proposition of a candidate Lyapunov function becomes more intricate. In this regard, the implicit Lyapunov function approach has proved to be useful [27, 28]. This approach, *encapsulates* the conditions that an implicitly defined Lyapunov function has to satisfy in order to assert finite-time or fixed-time stability. Thus, explicit knowledge of a Lyapunov function is avoided and substituted by a set of conditions that an implicitly defined Lyapunov function has to satisfy. This approach also allows to obtain convergence time estimates, which even if conservative, provide a rigorous estimation of the maximum convergence time [27]. In the same line of thought, the benefits of the implicit Lyapunov approach can be extended to the study of asymptotic and non-asymptotic ISS.

The contributions of this article, for which a preliminary version can be found in [29], can be summarized as follows:

- A complete framework (with necessary and sufficient conditions) to study the finite-time ISS property using Lyapunov analysis.
- A sufficient condition for fixed-time ISS stability.
- An extension of the explicit Lyapunov framework to the implicit approach.
- Alternative dissipativity-like formulations of the results. In some cases, this alternative formulation involves more tractable terms, making the conditions easier to verify.

Proofs of all theorems, corollaries and propositions are presented in the appendix.

1.1. Notation

- $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.
- The notation $DV(x)f(x)$ stands for the directional derivative of the continuously differentiable function V with respect to the vector field f evaluated at point x. If the function V is not continuously differentiable at x, then $DV(x)f(x)$ stands for the upper-right Dini derivative of V evaluated at x.
- For a continuous function $\rho(x, y, ...)$, $\partial_x \rho(x, y, ...)$ represents the partial derivative of ρ with respect to x; when the context allows no ambiguity the function arguments may be omitted, *i.e.* $\partial_x \rho$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, we use $||d||_{[t_0,t_1)} = \text{ess sup}_{t\in[t_0,t_1)}\|d(t)\|$ to define the norm of $d(t)$ in the interval $[t_0, t_1)$, then $||d||_\infty = ||d||_{[0,+\infty)}$ and the set of essentially bounded and measurable functions $d(t)$ with the property $||d||_\infty < +\infty$ is further denoted as $\mathcal{L}_\infty$; $\mathcal{L}_D = \{d \in \mathcal{L}_\infty : ||d||_\infty \leq D\}$ for any $D > 0$.
- For any $r \in \mathbb{R}$, the notation $\lceil\cdot\rfloor^r$ is an abbreviation of $|\cdot|^r \operatorname{sign}(\cdot)$.

2. Preliminaries

Consider the following nonlinear system

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the input, $d \in \mathcal{L}_\infty$; $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuous, ensures forward existence of the system solutions, at least locally, and $f(0, 0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and an input $d \in \mathcal{L}_\infty$, define the corresponding solution of (1) by $X(t, x_0, d)$ for any $t \geq 0$ for which the solution exists. Since (1) might not have unique solutions and in this work we are interested in the strong stability notions only (satisfying for all solutions), then with a slight inexactness in the notation we will assume that if a property is satisfied for all initial conditions in a set, then it implies that it also holds for all solutions issued from those initial conditions.

2.1. Stability Rates

Following [1, 9, 30], let $\Gamma \subseteq \mathbb{R}^n$ be an open connected set containing the origin.

Definition 1. The origin of the system (1), for $d \in \mathcal{L}_D$ and $D > 0$, is said to be

uniformly Lyapunov stable if for any $x_0 \in \Gamma$ and $d \in \mathcal{L}_D$ the solution $X(t, x_0, d)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x_0 \in \Gamma$, if $\|x_0\| \leq \delta$ then $\|X(t, x_0, d)\| \leq \epsilon$ for all $t \geq 0$;

uniformly asymptotically stable if it is uniformly Lyapunov stable and for any $\kappa > 0$ and $\epsilon > 0$ there exists $T(\kappa, \epsilon) \geq 0$ such that for any $x_0 \in \Gamma$ and $d \in \mathcal{L}_D$, if $\|x_0\| \leq \kappa$ then $\|X(t, x_0, d)\| \leq \epsilon$ for all $t \geq T(\kappa, \epsilon)$;

finite-time stable if it is uniformly Lyapunov stable and ***finite-time converging from*** Γ, *i.e.* for any $x_0 \in \Gamma$ and all $d \in \mathcal{L}_D$ there exists some constant $T \in \mathbb{R}_{\geq 0}$ such that $X(t, x_0, d) = 0$ for all $t \geq T$. The function $T(x_0) = \inf\{T \geq 0 : X(t, x_0, d) = 0 \ \forall t \geq T, \ \forall d \in \mathcal{L}_D\}$ is called the ***settling-time function*** of the system (1);

fixed-time stable if it is finite-time stable and $\sup_{x_0 \in \Gamma} T_0(x_0) < +\infty$.

The set Γ is called the ***domain of attraction*** and throughout the paper will be considered to be $\Gamma = \mathbb{R}^n$.

2.2. Comparison Functions

A continuous function $\vartheta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a ***class-$\mathcal{K}$ function*** if it is strictly increasing with $\vartheta(0) = 0$; ϑ is a ***class-$\mathcal{K}_\infty$ function*** if it is a class-$\mathcal{K}$ function and $\vartheta(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class-$\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is decreasing and $\lim_{t \to +\infty} \beta(s, t) = 0$ for each fixed $s \in \mathbb{R}_{\geq 0}$.

2.3. Generalized Comparison Functions

In order to define the property of non-asymptotic ISS, *conventional* class-$\mathcal{KL}$ functions are no longer suitable, therefore, a generalization of these functions is proposed.

Definition 2. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *generalized class-$\mathcal{KL}$ function* ($\mathcal{GKL}$ function) if

i) the mapping $s \mapsto \beta(s, 0)$ is a class-$\mathcal{K}$ function;

ii) for each fixed $s \geq 0$ the mapping $t \mapsto \beta(s, t)$ is continuous, decreases to zero and there exists some $\widetilde{T}(s) \in [0, +\infty)$ such that $\beta(s, t) = 0$ for all $t \geq \widetilde{T}(s)$.

Compared to $\mathcal{KL}$ functions, a $\mathcal{GKL}$ function has to be a $\mathcal{K}$ function only for $t = 0$ whereas a $\mathcal{KL}$ function has to be so for any fixed $t \geq 0$. Moreover, a $\mathcal{GKL}$ function not only has to be continuous and decreasing for each fixed s, but also has to converge to zero in a finite amount of time. Note that the definition of a $\mathcal{GKL}$ function presented here differs from the one introduced in [25].

Asymptotic Input-to-state Stability.

Definition 3 ([3])**.** The system (1) is called ***input-to-state stable (ISS)*** if for any input $d \in \mathcal{L}_\infty$ and any $x_0 \in \mathbb{R}^n$ there exist some functions $\beta \in \mathcal{KL}$ and $\vartheta \in \mathcal{K}$ such that

$$\|X(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \vartheta(||d||_{[0,t)}) \quad \forall t \geq 0.$$

The function ϑ is called the ***nonlinear gain***.

Non-Asymptotic Input-to-State Stability.

Definition 4. System (1) is said to be ***finite-time ISS*** (FTISS) if for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty$, each solution $X(t, x_0, d)$ is defined for $t \geq 0$ and satisfies

$$\|X(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \vartheta(\|d\|_\infty), \tag{2}$$

where ϑ is a class-$\mathcal{K}$ function and β is a class-$\mathcal{GKL}$ function with $\beta(r, t) = 0$ when $t \geq \widetilde{T}(r)$ with $\widetilde{T}(r)$ continuous with respect to r and $\widetilde{T}(0) = 0$. If, furthermore, $\sup_{r \in \mathbb{R}_{\geq 0}} \widetilde{T}(r) < +\infty$, the system (1) is said to be ***fixed-time ISS*** (FXISS).

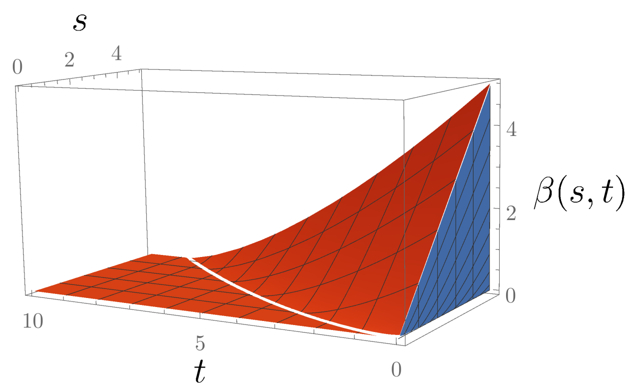

Figure 1: 3D plot of the function $\beta(s,t)$ of Example 1.

Remark that indeed the key difference with respect to asymptotic ISS is that β is a $\mathcal{GKL}$ function and that according to Definition 2 this implies the existence of the settling-time function $\widetilde{T}$. Remark also that only the case of continuous $\widetilde{T}$ is considered.

Example 1. The trajectories of the input scalar system

$$\dot{x} = -\lceil x\rfloor^{\frac{1}{3}} + d^2 \tag{3}$$

satisfy, for all $x, d \in \mathbb{R}$, the inequality (2) with

$$\beta(s,t) = \begin{cases} (s^{\frac{2}{3}} - \frac{1}{3}t)^{\frac{3}{2}} & \text{if } 0 \leq t \leq \widetilde{T}(s) \\ 0 & \text{otherwise} \end{cases}, \quad \widetilde{T}\,(s) = 3s^{\frac{2}{3}}, \quad \vartheta(s) = 2s^2.$$

Since $\beta(s,0) = s$ is a class-$\mathcal{K}$ function, $\beta(s,t) = 0$ for all $t \geq \widetilde{T}(s)$ and for each fixed s, $\beta(s,t)$ is decreasing (see Figure 1), β is a class-$\mathcal{GKL}$ function and the system (3) is FTISS. Figure 1 also shows, in blue, the projection of $\beta(s,0)$ on the s-β axis. The white line on the t-s axis represents the settling-time curve $t = 3s^{\frac{2}{3}}$.

The plot of the function β in Figure 1 helps to understand why class-$\mathcal{GKL}$ functions are needed to define non-asymptotic ISS: by looking at the grid lines on the s axis, it is possible to notice that for any fixed $t \neq 0$, $\beta(s,t)$ is not a class-$\mathcal{K}$ function since it equals to zero for multiple values of s. Hence, the FTISS property cannot be defined using $\mathcal{KL}$ functions.

From the definition of FTISS, it follows that if system (1) is FTISS, for $d = 0$ it becomes finite-time stable with some continuous settling-time function T. Therefore, for FTISS systems, the existence of $T(x)$ implies that of $\widetilde{T}(x)$ and vice versa. Hence FTISS implies finite-time stability when $d = 0$, however, as the next example shows, the converse is in general not true.

Example 2. The state of the system

$$\dot{x} = -(1 + \sin d)\lceil x\rfloor^{\frac{1}{3}} \tag{4}$$

is bounded for each bounded input $d \in \mathbb{R}$. Moreover, for $d = 0$ the origin of (4) is FTS. However, for $d = 3\pi/2$, the origin is not even asymptotically stable and therefore not FTISS.

2.4. Asymptotic ISS Lyapunov functions

Definition 5 ([2]). A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called ***ISS Lyapunov function*** for system (1) if for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\chi, \gamma \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{5}$$

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -\gamma(\|x\|). \tag{6}$$

Remark that, without loss of generality, one can assume that $\gamma \in \mathcal{K}_\infty$ [31, Remark 4.1].

As the next lemma states, there exists an alternative definition of an ISS Lyapunov function that provides a *dissipativity like* characterization, *i.e.*, it includes a dissipation term that eventually vanishes, leaving only an input-dependent term. Depending on the family and context of the systems under study, this representation may be more relevant, ease the calculations or render the equations more tractable [3].

Lemma 1 ([2]). *A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS Lyapunov function for* (1) *if and only if there exist $\alpha_1, \alpha_2, \delta, \zeta \in \mathcal{K}_\infty$ such that* (5) *holds and*

$$DV(x)f(x,d) \leq \delta(\|d\|) - \zeta(\|x\|), \quad \text{for all } x \in \mathbb{R}^n \text{ and } d \in \mathbb{R}^m. \tag{7}$$

The following theorem is the main result in ISS theory, it relates the existence of an ISS Lyapunov function with the ISS property of a given system.

Theorem 1 ([2]). *The system (1) is ISS if and only if it admits an ISS Lyapunov function.*

2.5. Implicit Lyapunov Functions

The implicit Lyapunov function approach combines the implicit function theorem with Lyapunov's direct method in order to determine the conditions that a function $Q(V,x)$ has to satisfy in order to implicitly define, through the solution of the equation $Q(V,x) = 0$, a Lyapunov function V.

Theorem 2 ([27]). *If there exists a continuous function*

$$\begin{aligned} Q : \mathbb{R}_+ \times \mathbb{R}^n &\to \mathbb{R}, \\ (V,x) &\to Q(V,x) \end{aligned}$$

satisfying the conditions

C1 *Q is continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$;*

C2 *for any $x \in \mathbb{R}^n \backslash \{0\}$ there exists $V \in \mathbb{R}_+$: $Q(V,x) = 0$;*

C3 *for $\Omega = \left\{(V,x) \in \mathbb{R}^{n+1} : Q(V,x) = 0\right\}$ we have* $\lim\limits_{\substack{x \to 0 \\ (V,x) \in \Omega}} V = 0$, $\lim\limits_{\substack{V \to 0^+ \\ (V,x) \in \Omega}} \|x\| = 0$, $\lim\limits_{\substack{\|x\| \to \infty \\ (V,x) \in \Omega}} V = +\infty$;

C4 *$-\infty < \partial_V Q(V,x) < 0$ for $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \backslash \{0\}$;*

C5 *$\partial_x Q(V,x) f(x,0) < 0$ for all $(V,x) \in \Omega$,*

then the origin of (1) *with $d = 0$ is globally uniformly asymptotically stable and the function $Q(V,x) = 0$ implicitly defines a Lyapunov function $V(x)$ for* (1).

Conditions **C1** and **C4** are required to satisfy the implicit function theorem. Conditions **C2** and **C3** ensure that $Q(V,x) = 0$ defines implicitly a unique, continuously differentiable, radially unbounded and positive definite function V. The last condition not only implies that V satisfies the differential inequality of Lyapunov's direct method, it also determines the system's convergence rate. Indeed, as shown in [27], if conditions **C1** and **C4** hold and $\partial_x Q f(x,0) < c_1 V^a \partial_V Q$, where $c_1 > 0$ and $a \in [0,1)$ the origin is finite-time stable; if $\partial_x Q f(x,0) < (c_1 V^a + c_2 V^b) \partial_V Q$, where c_1 and a are as before, $c_2 > 0$ and $b > 1$, the origin is fixed-time stable.

3. Finite-Time and Fixed-Time ISS Lyapunov Functions: Explicit Approach

3.1. Finite-Time ISS Lyapunov Functions

Definition 6. Consider a positive definite and radially unbounded C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. V is called a ***finite-time ISS Lyapunov function*** for system (1) if there exist some $\chi \in \mathcal{K}$, $c_1 > 0$ and $a \in [0,1)$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -c_1[V(x)]^a. \tag{8}$$

In [25], some sufficient conditions for finite-time ISS with continuous settling-time function are presented. However converse results are not obtained. The following results show that if some assumptions on the Lipschitz continuity of the system outside the origin and of the settling-time function are added, then a converse result exists.

Assumption 1. Let on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^m$ the function $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be locally Lipschitz continuous and, in addition, there exists a continuous function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\|f(x,d) - f(x,0)\| \leq L(\|x\|)\|d\|$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.

Theorem 3. *The system* (1) *is FTISS if it admits a finite-time ISS Lyapunov function. Conversely, if* (1) *is FTISS with a locally Lipschitz continuous settling-time function and Assumption 1 is satisfied, then there exists a finite-time ISS Lyapunov function for it.*

Regarding the settling-time estimate when perturbations are absent, the following corollary can be straightforwardly obtained from previous results

Corollary 1. *Assume that any of the conditions of Theorem 3 holds, then the settling-time function T satisfies*

$$T(x) \leq \frac{1}{c_1(1-a)}[V(x)]^{1-a} \tag{9}$$

for $d = 0$ and for all $x \in \mathbb{R}$.

Example 3 (*ISS vs FTISS*)**.** Consider the input systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -x_1 - x_2 + d_1 \\ \dot{x}_2 = x_1 - x_2^3 + d_2 \end{cases}, \quad \Sigma_2 : \begin{cases} \dot{x}_1 = -\lceil x_1 \rfloor^\gamma - x_2 + d_1 \\ \dot{x}_2 = x_1 - \lceil x_2 \rfloor^\gamma + d_2 \end{cases}, \quad x, d \in \mathbb{R}^2, \gamma \in (0,1)$$

and the ISS Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Note that

$$x_1^2 \leq x_1^2 + x_2^4, \ |x_1| + x_2^2 \leq \sqrt{2}\sqrt{x_1^2 + x_2^4},$$

where Jensen's inequality was used for the second relation, then we have

$$\begin{aligned} \|x\|^2 &= x_1^2 + x_2^2 \leq \left(x_1^2 + x_2^4\right) + \sqrt{2}\sqrt{x_1^2 + x_2^4} \\ &= \alpha(x_1^2 + x_2^4) \end{aligned}$$

for the class $\mathcal{K}_\infty$ function $\alpha(s) = s + \sqrt{2s}$. Hence,

$$DV(x)f(x,d)\Big|_{\Sigma_1} \leq -x_1^2 - x_2^4 + 2\|x\|\|d\| \leq -\tilde{\alpha}(\|x\|) + 2\|x\|\|d\|$$

where $\tilde{\alpha}(s) = \alpha^{-1}(s^2) = s^2 + 1 - \sqrt{2s^2 + 1}$. A direct analysis shows that

$$\tilde{\alpha}(s) \geq \frac{1}{4}\min\{s^4, s^2\},$$

therefore,

$$\|d\| \leq \frac{1}{16}\min\{\|x\|^4, \|x\|^2\} \Rightarrow DV(x)f(x,d) \leq -\frac{1}{2}\tilde{\alpha}(\|x\|)$$

and the system is ISS.

Proceeding similarly for Σ_2, we have that

$$\begin{aligned} DV(x)f(x,d)\Big|_{\Sigma_2} &= -|x_1|^{1+\gamma} - |x_2|^{1+\gamma} + x^\top d \\ &\leq -|x_1|^{1+\gamma} - |x_2|^{1+\gamma} + \|x\|\|d\|, \end{aligned}$$

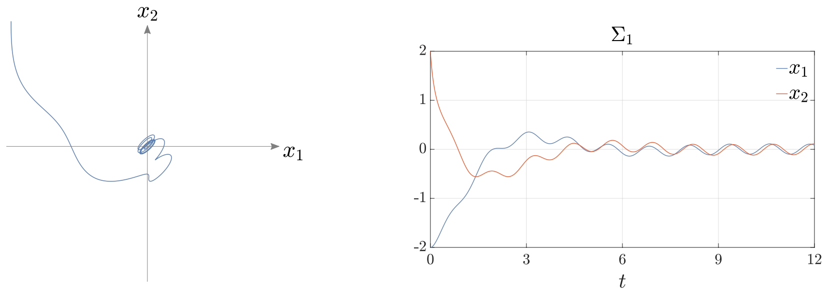

Figure 2: Phase space diagram (left) and time plot (right) of a trajectory of the ISS system Σ_1 with initial condition $x_0 = (-2, 2)$ and $d_1(t) = d_2(t) = 0.5\sin(5t)$.The effect of the perturbation prevents the trajectories from reaching the origin, however they remain confined in a vecinity of the origin.

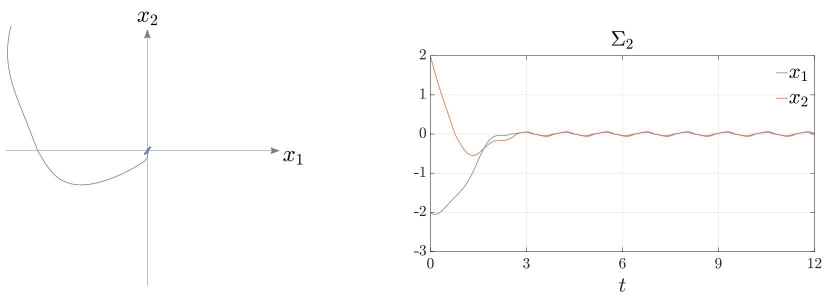

Figure 3: Phase space diagram (left) and time plot (right) of a trajectory of the FTISS system Σ_2 with initial condition $x_0 = (-2, 2)$ and $d_1(t) = d_2(t) = 0.5\sin(5t)$. Compared to ISS, the trajectories of the FTISS system Σ_2 remain confined in a smaller vicinity of the origin, which shows a better degree of robustness.

since

$$\left(x_1^2 + x_2^2\right)^{\frac{1+\gamma}{2}} \leq |x_1|^{1+\gamma} + |x_2|^{1+\gamma}$$

for all $x_1, x_2 \in \mathbb{R}$, then

$$DV(x)f(x,d)|_{\Sigma_2} \leq -\|x\|^{1+\gamma} + \|x\|\|d\|$$

and for $\|d\| \leq \frac{1}{2}\|x\|^{\gamma}$ we obtain

$$DV(x)f(x,d)\Big|_{\Sigma_2} \leq -\frac{1}{2}\|x\|^{1+\gamma} = -2^{\frac{\gamma-1}{2}} V^{\frac{1+\gamma}{2}}(x).$$

Then, according to Theorem 3, Σ_2 is an FTISS system. Figure 2 shows a trajectory of Σ_1 starting at $x_0 = (-2, 2)$ with inputs $d_1(r) = d_2(t) = 0.5\sin(5t)$. Figure 3 shows a trajectory of Σ_2 starting at the same initial condition and with the same inputs. It becomes noticeable that the trajectories of Σ_2 are less influenced by the disturbance d and that they remain contained in a smaller vicinity of the origin. Using Corollary 1, the settling-time function for $d_1 = d_2 = 0$ can be estimated as

$$T(x) \leq \tfrac{4}{(1-\theta)(1-\gamma)}(x_1^2 + x_2^2)^{\frac{\gamma+1}{2}}.$$

3.2. Fixed-Time ISS Lyapunov Functions

Definition 7. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as defined in Definition 6 that additionally satisfies

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -c_1[V(x)]^a - c_2[V(x)]^b \quad \forall x \in \mathbb{R}^n, d \in \mathbb{R}^m, \tag{10}$$

where $c_2 > 0$ and $b > 1$, is called a ***fixed-time ISS Lyapunov function***.

Theorem 4. *The system* (1) *is FXISS if it admits a fixed-time ISS Lyapunov function.*

Corollary 2. *If the Theorem 4 holds and $d = 0$, then the settling-time function T satisfies*

$$T(x) \leq \frac{1}{c_1(1-a)} + \frac{1}{c_2(b-1)} \quad \forall x \in \mathbb{R}^n. \tag{11}$$

Note that this bound on the settling-time is independent of x. The lack of a converse result on FXISS stems from the fact that a necessary conditions for fixed-time stability with continuous settling-time function in autonomous systems is still an open problem [10]. In [10], we discussed a property called *complete* fixed-time stability, for which converse results were obtained. Although these results may be extended for input systems, this exceeds the scope of this article.

4. Finite-Time and Fixed-Time ISS Lyapunov Functions: Implicit Approach

This section introduces the tools to determine the FTISS and FXISS properties using the Implicit Lyapunov approach. Let us insist on the fact that this approach allows to circumvent explicit knowledge of a Lyapunov function V and its derivative; instead, knowledge of an implicit function $Q(V, x)$ and its derivative is required. This new conditions on the function Q may be more difficult to verify in the general case; however, as shown in Example 5, a combination with a suitable proposition of Q can ease the calculations and succeed in determining non-asymptotic ISS properties where the explicit approach fails to do so.

Definition 8. A continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is called an ***implicit ISS Lyapunov function*** for system (1) if it satisfies all conditions of Theorem 2 for $d = 0$ and

C5$^{\text{iss}}$ $\|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \gamma(\|x\|)$

for all $(V, x) \in \Omega$ and all $d \in \mathbb{R}^m$, with $\chi, \gamma \in \mathcal{K}$.

Theorem 5. *System* (1) *is ISS if and only if there exists an implicit ISS Lyapunov function $Q(V, x)$ for it.*

4.1. Implicit Finite-Time and Fixed-Time ISS Lyapunov Functions

Definition 9. Consider a continuous function $Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ that satisfies all conditions of Theorem 2 for $d = 0$. Q is called an ***implicit finite-time ISS Lyapunov function*** for (1) if there exist some $\chi \in \mathcal{K}$, $c_1 > 0$ and $a \in [0, 1)$ such that for all $(V, x) \in \Omega$ and all $d \in \mathbb{R}^m$

C5$^{\text{ft}}$ $\|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^a.$

Q is called an ***implicit fixed-time ISS Lyapunov Function*** for (1) if there exist some $\chi \in \mathcal{K}$, $c_1, c_2 > 0$, $a \in [0, 1)$ and $b > 1$ such that for all $(V, x) \in \Omega$ and all $d \in \mathbb{R}^m$

C5$^{\text{fx}}$ $\|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^a + c_2 V^b.$

Theorem 6. *System* (1) *is FTISS if there exists an implicit finite-time ISS Lyapunov function for it. Conversely, if* (1) *is FTISS with a locally Lipschitz continuous settling-time function and Assumption 1 is satisfied, then there exists an implicit finite-time ISS Lyapunov function for it. The settling-time estimate for $d = 0$ and for all $x_0 \in \mathbb{R}^n$ satisfies $T(x_0) \leq \frac{1}{c_1(1-a)} V_0^a$ for some $c_1 > 0$, where $V_0 \in \mathbb{R}_{\geq 0} : Q(V_0, x_0) = 0$.*

Theorem 7. *System* (1) *is FXISS if there exists an implicit fixed-time ISS Lyapunov function for it.*

Remark that since in the case of FXISS, the settling-time estimate is independent of the initial conditions, the estimate (11) remains valid for the implicit approach.

Example 4. Consider the system

$$\dot{x} = -x^3 - x^{\frac{1}{3}} + x^2 d_1 - x d_2 + d_1 d_2 \tag{12}$$

and the following implicit ISS Lyapunov function candidate:

$$Q(V, x) = \frac{x^2}{2V} - 1. \tag{13}$$

We have that $\frac{\partial Q(V,x)}{\partial V} = -\frac{x^2}{2V^2}$, and that $\frac{\partial Q(V,x)}{\partial x} = \frac{x}{V}$, hence

$$-\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) = -\frac{Vx}{2x^2}(-x^3 - x^{\frac{1}{3}} + x^2 d_1 - x d_2 + d_1 d_2),$$

if $3|d_1| \leq |x|$ and $3|d_2| \leq x^2$ and since $Q = 0 \Rightarrow 1 = \frac{x^2}{2V}$ we have

$$\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \frac{2}{9}x^4 + \frac{1}{4}x^{\frac{5}{3}}.$$

Then, according to Theorem 7 the function $Q(V,x)$ defined by (13) is an implicit fixed-time ISS Lyapunov function for (12) with $\gamma(|x|) = \frac{2}{9}x^4 + \frac{1}{4}x^{\frac{5}{3}}$ and $\chi(\bar{d}) = \nu^{-1}(\bar{d})$, $\nu(r) = \min\{\frac{r}{3}, \frac{r^2}{3}\}$, $\bar{d} = \max\{d_1, d_2\}$ and we conclude that the origin of (12) is FXISS. Note that although in this example it is possible to obtain an explicit expression for V, using the implicit framework this is no longer necessary and FXISS can be determined using information about Q only.

Example 5. Consider the double integrator

$$\dot{x} = A_0 x + bu(x) + d, \tag{14}$$
$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ d \in \mathbb{R}^2,$$

with the following control law

$$u(x) = -k_1 \lceil x_1 \rfloor^{\frac{\nu}{2-\nu}} - k_2 \lceil x_2 \rfloor^{\nu}, \tag{15}$$

where $k_1, k_2 > 0$ and $\nu \in (0,1)$.

In [32], instead of using an explicit Lyapunov function, finite-time stability of this system, for $x \in \mathbb{R}^n$ and $d = 0$, is proven by first considering the asymptotic stability of (14)-(15) when $\nu = 1$. In this case, the closed loop system becomes linear and it is possible to propose a quadratic Lyapunov function

$$V(x) = x^T P x, \tag{16}$$

$\partial_x V(x) f(x,0) < 0$ for all $x \in \mathbb{R}^2 \backslash \{0\}$ and for properly selected $k = (k_1, k_2)$. Next the authors show that $\{x : V(x) \leq 1\}$ is strictly positively invariant under (14)-(15), for ν sufficiently close to 1, and using homogeneity arguments (see [32, Theorem 6.1]), global finite-time stability and ISS are obtained. In this example we will show that it is possible to construct an implicit FT ISS Lyapunov function for (14)-(15), provided that ν is sufficiently close to 1.

Let us propose the following implicit Lyapunov function candidate[1]:

$$Q(V,x) = x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \tag{17}$$

where $P \in \mathbb{R}^{2\times 2}, P > 0$ and $D_r(V^{-1}) = \left(\begin{smallmatrix} V^{-r_1} & 0 \\ 0 & V^{-r_2} \end{smallmatrix}\right)$, $r_1 = \frac{2-\nu}{2}, r_2 = \frac{1}{2}$. The function $Q(V,x)$ is differentiable for any $(V,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$ and since $P > 0$ then

$$\frac{\lambda_{\min}(P)\|x\|^2}{V} \leq Q(V,x) + 1 \leq \frac{\lambda_{\max}(P)\|x\|^2}{V^{2-\nu}}, \tag{18}$$

and there exist some $V^-, V^+ \in \mathbb{R}_+$ such that $Q(V^-,x) < 0 < Q(V^+,x)$ and some $V \in \mathbb{R}_+$ such that $Q(V,x) = 0$. Hence conditions **C1-C3** are fulfilled. Remark that for $\nu = 1$, the identity $Q(V,x) = 0$ defines the quadratic Lyapunov function (16).

The derivative of Q w.r.t. V is given by

$$\partial_V Q = -V^{-1} x^T D_r(V^{-1})(H_r P + P H_r) D_r(V^{-1}) x,$$

where $H_r = \left(\begin{smallmatrix} r_1 & 0 \\ 0 & r_2 \end{smallmatrix}\right)$. Since $H_r = \frac{1}{2}I_2 + \frac{1-\nu}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ where I_2 is the identity matrix, $H_r \to \frac{1}{2}I_2$ as $\nu \to 1$ and

$$0 < PH_r + H_r P,$$

[1] The proposed implicit Lyapunov function is discussed in detail in [27, 33].

so that $\partial_V Q(V,x) < 0$ for all $(V,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$ and condition **C4** is satisfied. Assuming additionally that $PH_r + H_r P \leq P$ and taking into account (from (17)) that $Q(V,x) = 0 \Rightarrow x^T D_r(\frac{1}{V}) P D_r(\frac{1}{V}) x = 1$ we obtain

$$-V^{-1} \leq \partial_V Q(V,x) < 0. \tag{19}$$

Similarly, the derivative of Q along the trajectories of (14)-(15), denoted as $\partial_x Q f$, is given by

$$\partial_x Q f = 2x^T D_r(\tfrac{1}{V}) P D_r(\tfrac{1}{V})(A_0 x + bu(x) + d).$$

Let us assume that the following condition holds for some $\mu > 1$:

$$A_0 S + S A_0^T + bq + b^T q^T + S + \mu I_2 \leq 0, \tag{20}$$

where $S = P^{-1}$ and $q = kS^{-1}$. By adding and subtracting the term $2V^{\nu/2} x^T D_r(\frac{1}{V}) P D_r(\frac{1}{V}) bk D_r(\frac{1}{V}) x$, and taking into account that $D_r(\frac{1}{V}) A_0 D_r^{-1}(\frac{1}{V}) = V^{(\nu-1)/2} A_0$ and that $D_r(\frac{1}{V}) b = V^{-\frac{1}{2}} b$, we obtain

$$\partial_x Q f = \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)^T \Theta \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) + V^{\frac{\nu-1}{2}}(2y^T P bk \tilde{y}_\nu - y^T P y) + \tfrac{1}{\mu} V^{\frac{1-\nu}{2}} z^T z,$$

where $y = D_r(\frac{1}{V})x$, $z = D_r(\frac{1}{V})d$, $\tilde{y}_\nu = y - (\lceil y_1 \rfloor^{\frac{\nu}{2-\nu}}, \lceil y_2 \rfloor^{\nu})^T$ and

$$\Theta = \begin{pmatrix} V^{\frac{\nu-1}{2}}(P(A_0 - bk) + (A_0 - bk)^T P + P) & P \\ P & -\frac{1}{\mu} V^{\frac{1-\nu}{2}} I_2 \end{pmatrix}. \tag{21}$$

Since the Schur complement of $\left(\begin{smallmatrix} P^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \Theta \left(\begin{smallmatrix} P^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is equivalent to the left hand side of (20) and $Q(V,x) = 0 \Rightarrow y^T P y = 1$, we have that

$$\partial_x Q f \leq V^{\frac{\nu-1}{2}}(2y^T P bk \tilde{y}_\nu - 1) + \tfrac{1}{\mu} V^{\frac{1-\nu}{2}} d^T D_r^2(\tfrac{1}{V}) d.$$

Since $\tilde{y}_\nu \to 0$ and $D_r^2(\frac{1}{V}) \to V^{-1} I_2$ as $\nu \to 1$, there exists some ν, sufficiently close to one, such that $\max_{y: y^T P y = 1} y^T P bk \tilde{y}_\nu < \frac{1}{\mu} < c_1 < 1$. Then

$$\partial_x Q < -c_2 V^{\frac{\nu-1}{2}} + \tfrac{1}{\mu} V^{-1+\frac{1-\nu}{2}} d^T d,$$

where $c_2 = 1 - c_1 > 0$. From (19) we obtain

$$\frac{\partial_x Q}{\partial_V Q} f(x,d) \geq c_1 V^{1+\frac{\nu-1}{2}} - \tfrac{1}{\mu} V^{-1+\frac{1-\nu}{2}} d^T d,$$

and from (18) we finally derive

$$\|x\| \geq \chi(\|d\|) \Rightarrow \frac{\partial_x Q}{\partial_V Q} f(x,d) \geq (c_1 - \tfrac{1}{\mu}) V^{1+\frac{\nu-1}{2}}$$

where $\chi(r) = \frac{1}{\lambda_{\min}(P)} r^{\frac{1}{\nu+1}}$ and we recover the condition $\mathbf{C5}^{\mathrm{ft}}$. Thus, we conclude that $Q(V,x)$ is a finite-time implicit Lyapunov function and from Theorem 6, the origin of (14)-(15) is FTISS for any ν sufficiently close to 1. Figure 4, shows the simulation plot of system (14) with the disturbance term $d_1(t) = d_2(t) = \hat{d}(t)$, where

$$\hat{d}(t) = 0.2\sin(10t) + \begin{cases} 1 & \text{if } t \in [5,6] \\ 0 & \text{otherwise} \end{cases}. \tag{22}$$

This example focuses in showing that it is always possible to find ν, sufficiently close to 1, such that (14)-(15) is FTISS; note that from (21) it is clear that it is always possible to find some P, k such that (20) holds. In a different scenario, if k and P are given, then it is possible to explicitly calculate the restrictions on ν.

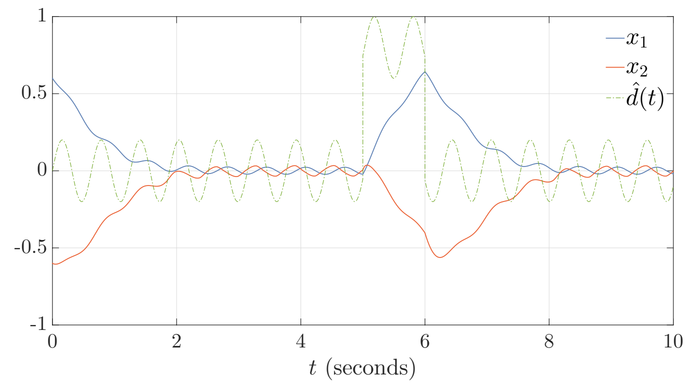

Figure 4: Simulation of system (14) for $\nu = 0.8$, initial conditions $x(0) = (0.6, -0.6)$, $k = (5, 5)$ and in dotted green the disturbance term $\hat{d}$. It is possible to see that the trajectories remain in a small vecinity of the origin in spite of the perturbation.

5. Dissipativity-like Formulation

Using the arguments of Lemma 1, it is possible to show that the following proposition is an equivalent definition of an implicit finite-time ISS Lyapunov function.

Proposition 1. *Suppose that there exists a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ that satisfies conditions* **C1**-**C4** *of Theorem 2 for $d = 0$ and*

$$\textbf{C5}^{\text{iss}*} \quad \frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \zeta(\|x\|) - \delta(\|d\|)$$

for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$, where $\delta, \zeta \in \mathcal{K}_\infty$. Then (1) *is ISS and $Q(V,x)$ is an implicit ISS Lyapunov function for* (1).

With a mild modification of the proof of Theorem 6, we obtain the next proposition.

Proposition 2. *Consider a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ that satisfies conditions* **C1**-**C4** *for $d = 0$. If there exist some $\delta \in \mathcal{K}_\infty$, $c_1 > 0$ and $a \in [0,1)$ such that*

$$\textbf{C5}^{\text{ft}*} \quad \frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^a - \delta(\|d\|)$$

for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$, then (1) *is FTISS and $Q(V,x)$ is an implicit finite-time ISS Lyapunov function for* (1). *If there exist some $\delta \in \mathcal{K}_\infty$, $c_1, c_2 > 0$, $a \in [0,1)$ and $b > 1$ such that*

$$\textbf{C5}^{\text{fx}*} \quad \frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^a + c_2 V^b - \delta(\|d\|)$$

for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$, then (1) *is FXISS and $Q(V,x)$ is an implicit fixed-time ISS Lyapunov function for* (1).

6. Conclusions

A theoretical framework, with necessary and sufficient conditions, for implicit ISS and implicit finite-time ISS Lyapunov functions has been developed. For the implicit fixed-time ISS case, sufficient conditions were provided and a possible direction to obtain a converse result was discussed.

All the proposed theorems and definitions allow to assert, with a single implicitly defined function, the convergence type and the robustness, in an input-to-state sense, of a given nonlinear system. Also, with the results presented, it is possible to obtain the convergence time to zero whenever the disturbances are absent. In order to achieve these results, necessary and sufficient conditions for finite-time ISS explicit Lyapunov functions were developed for the first time.

The implicit framework, being developed from Lyapunov's direct method, also lacks a universal methodology to obtain implicit Lyapunov functions, however, as seen in Example 5, it can be used in systems where no explicit LF is known, thus broadening the Lyapunov analysis tools.

To derive nonlinear gains from a given implicit Lyapunov function is proposed as a future research topic.

7. Appendix

Proof of Theorem 3.

Sufficiency. If there exists a finite-time ISS Lyapunov function for (1), then we have that $\|x\| \geq \chi(\|d\|)$ implies that

$$DV(x)f(x,d) \leq -c_1[V(x)]^{\alpha} \tag{23}$$

and from Definition 5 we know that (5) holds.

I Let us define the set $\mathcal{V} = \{x : V(x) \geq \alpha_2 \circ \chi(\|d\|_{[0,\infty)})\}$. We have that for any $x \in \mathcal{V}$, $\alpha_2(\|x\|) \geq V(x) \geq \alpha_2 \circ \chi(\|d\|_{\infty})$, which implies that $\|x\| \geq \chi(\|d\|)$ and by (23), $\mathbb{R}^n \backslash \mathcal{V}$ is an invariant and attractive set. Then, using the comparison lemma and direct integration, it is straightforward to obtain a class-$\mathcal{GKL}$ function $\beta(r,t)$ such that

$$\|X(t,x_0,d)\| \leq \beta(\|x_0\|,t) \ \text{ while } X(t,x_0,d) \in \mathcal{V}, \tag{24}$$

where $\beta(r,t) = 0 \ \forall t \geq \widetilde{T}(r)$ and $\widetilde{T}(r)$ is a continuous function for all $r \in \mathbb{R}^n$.

II If $x \notin \mathcal{V}$ then $V(x) < \alpha_2 \circ \chi(\|d\|_{[0,\infty)})$ and therefore $\|x\| \leq \vartheta(\|d\|_{[0,\infty)})$, where $\vartheta = \alpha_1^{-1} \circ \alpha_2 \circ \chi$. In addition, $\mathbb{R}^n \backslash \mathcal{V}$ is invariant so that

$$\|X(t,x_0,d)\| \leq \vartheta(\|d\|_{[0,\infty)}) \ \text{ while } X(t,x_0,d) \notin \mathcal{V}, \tag{25}$$

IV Combining (24) and (25) gives

$$\|X(t,x_0,d)\| \leq \beta(\|x_0\|,t) + \vartheta(\|d\|_{[0,\infty)}) \quad \forall t \geq 0,$$

and FTISS for (1) is obtained.

Necessity. This part of the proof has four main steps. First, using converse arguments, a Lyapunov function $V(x)$ is constructed that shows FTS of the unperturbed system (1). Second, it is shown that this Lyapunov function $V(x)$ is actually an FTISS Lyapunov function if $\|x\| < \rho$, for any $\rho > 0$ (with the asymptotic gain dependent on ρ). Third, applying smoothing tools another ISS Lyapunov function $W(x)$ is designed for $\|x\| > \delta$ for any $\delta \in (0,\rho)$. Finally, a desired global FTISS Lyapunov function is constructed by uniting V and W.

I Since (1) is FTISS, when $d = 0$ there exists some $T(x)$ such that $\|X(t,x,0)\| = 0 \ \forall t \geq T(x)$. If $T(x)$ is a locally Lipschitz function, then by [4, Theorem 4.3] it is possible to define a function $V(x) := T(x)^{\frac{1}{1-a}}$, with[2] $a \in [0,1)$, satisfying $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and such that

$$DV(x)f(x,0) \leq -c_1[V(x)]^a \tag{26}$$

for some $c_1 > 0$ and for almost all $x \in \mathbb{R}^n$.

II (Case $\|x\| < \rho$). Since $T(x)$ is locally Lipschitz continuous, $V(x)$ is also locally Lipschitz continuous and $\|\frac{\partial V}{\partial x}\| \leq \kappa + \eta(\|x\|)$ for some $\kappa \in \mathbb{R}_{\geq 0}$ and $\eta \in \mathcal{K}$. By Assumption 1, $\|f(x,d) - f(x,0)\| \leq L(\|x\|)\|d\|$ for some $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Thus

$$\big\|DV(x)\left(f(x,d) - f(x,0)\right)\big\| \leq (\kappa + \eta(\|x\|))L(\|x\|)\|d\|$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$. Let us define

$$d := \varphi_\rho(\|x\|)u \ \text{ where } \ \varphi_\rho(\|x\|) := \frac{c\,\alpha_1(\|x\|)^a}{2(\kappa + \eta(\rho))L(\rho)}. \tag{27}$$

Then, for $\|u\| \leq 1$ and some $\rho > 0$, it becomes clear that $\|x\| \leq \rho$ implies

$$\big\|DV(x)\big(f(x,\varphi_\rho(\|x\|)u) - f(x,0)\big)\big\| \leq \tfrac{c}{2}[V(x)]^a.$$

From (27), it follows that $\|u\| \leq 1$ implies that $\|x\| \geq \varphi_\rho^{-1}(\|d\|)$ and using the inequality (26) we have that $\rho > \|x\| \geq \varphi_\rho^{-1}(\|d\|)$ implies

$$DV(x)f(x,d) \leq -\tfrac{c}{2}[V(x)]^a.$$

[2]The reference [4] considers the interval $a \in (0,1)$ only, an extenstion of the same result for the interval $a \in [0,1)$ can be found, for instantce, in [34].

III (Case $\|x\| > \delta$). For any two constants $L_x > 0$ and $L_u > 0$ define the function

$$\mu_\delta(v,d) := \min\left\{1, \frac{(L_x v + L_d\|d\|)(1+\sup_{\|x\|\le v}\|f(x,d)\|)}{(L_x \delta + L_d\|d\|)(1+\sup_{\|x\|\le \delta}\|f(x,d)\|)}\right\},$$

for some $\delta \in (0,\rho)$. Note that by design μ_δ is continuous, increasing, bounded by 1, equals to 1 when $v=\delta$, strictly positive outside of the origin and $\mu_\delta(0,0)=0$. Define a vector field

$$f_\delta(x,d) := \begin{cases} f(x,d), & \text{if } \|x\| \ge \delta \\ \mu_\delta(\|x\|,d) f(x,d), & \text{if } \|x\| < \delta \end{cases},$$

which is locally Lipschitz and continuous by construction. Indeed, the function f possesses this property outside of the origin by the imposed hypothesis, and for $\|x\| < \delta$ we have that

$$\begin{aligned}
\|f_\delta(x,d)\| &\le \|\mu_\delta(\|x\|,d)f(x,d)\| \\
&= \frac{\|f(x,d)\|}{1+\sup_{\|s\|\le\delta}\|f(s,d)\|} \cdot \frac{1+\sup_{\|s\|\le\|x\|}\|f(s,d)\|}{L_x\delta + L_d\|d\|}\left(L_x\|x\| + L_d\|d\|\right) \\
&\le \frac{1+\sup_{\|s\|\le\|x\|}\|f(s,d)\|}{L_x\delta + L_d\|d\|}\left(L_x\|x\| + L_d\|d\|\right) \\
&\le \frac{1+\sup_{\|s\|\le\delta}\|f(s,d)\|}{L_x\delta + L_d\|d\|}\left(L_x\|x\| + L_d\|d\|\right),
\end{aligned}$$

so that f_δ is locally Lipschitz for all $x \in \mathbb{R}^n$. Now let us consider the system

$$\dot{x} = f_\delta(x,d),$$

where f_δ is, as showed above, a locally Lipschitz continuous function and it is ISS since (1) has this property (multiplication by a continuous strictly positive function μ_δ does not influence the stability, it acts as a time re-scaling). Consider now the following modified version of the system (1):

$$\dot{x} = f_\delta(x,d) = f(x,d) + \Delta f,$$

where $\Delta f := f(x,d) - f_\delta(x,d)$, and by construction $\|x\| \ge \delta \Rightarrow \Delta f = 0$. Following the converse results on existence of ISS Lyapunov functions, there exists a continuously differentiable, positive definite and radially unbounded function $W : \mathbb{R}^n_{\ge 0} \to \mathbb{R}_{\ge 0}$, $\alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\|x\| \ge \sigma(\|d\|) \Rightarrow \tfrac{\partial W(x)}{\partial x} f_\delta(x,d) \le -\alpha_3(\|x\|),$$

then due to the properties of the auxiliary perturbation Δf:

$$\|x\| \ge \max\{\delta, \sigma(\|d\|)\} \Rightarrow DW(x)f(x,d) \le -\alpha_3(\|x\|).$$

IV Let us define the function

$$\widetilde{V}(x) := s(V(x))W(x) + (1 - s(V(x)))V(x),$$

where $s : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ satisfies

$$s(r) = \begin{cases} 1 & \text{if } r \ge \alpha_2(\rho) \\ 0 & \text{if } r \le \alpha_1(\delta) \end{cases}$$

and $\dot{s} = \frac{\partial s(r)}{\partial r} > 0$ for all $r \in (\alpha_1(\delta), \alpha_2(\rho))$. Assume that $V(x) \le W(x)$ for all $x \in \{x \in \mathbb{R}^n : \alpha_1(\delta) \le V(x) \le \alpha_2(\rho)\}$ (both functions $V(x)$ and $W(x)$ are continuous, positive definite and radially unbounded, then we can adopt such a hypothesis without being restrictive, since multiplying $W(x)$ by a constant we can always assure its fulfillment), then we have that

$$D\widetilde{V}(x)f(x,d) = s\, DW(x)f(x,d) + (1-s)DV(x)f(x,d) + \dot{s}\, DV(x)f(x,d)(W(x) - V(x)),$$

and gathering all the previous estimates, we arrive to

$$\|x\| \geq \chi(\|d\|) \Rightarrow D\widetilde{V}(x)f(x,d) \leq -\alpha_4(\|x\|), \tag{28}$$

where $\chi(r) := \max\{\sigma(r), \varphi_\rho^{-1}(r)\}$ and $\alpha_4 \in \mathcal{K}_\infty$ such that

$$\alpha_4(\|x\|) \geq \begin{cases} \alpha_3(\|x\|), & V(x) \geq \alpha_2(\rho) \\ \frac{c}{2}[\widetilde{V}(x)]^a, & V(x) \leq \alpha_1(\delta) \end{cases}.$$

Consequently, $\widetilde{V}$ is a finite-time ISS Lyapunov function for (1). □

Proof of Corollary 1. This corollary is an immediate consequence of Theorems 4.2 and 4.3 of [4]. □

Proof of Theorem 4. The proof follows closely the reasoning of the sufficiency proof of Theorem 3. Instead of (26), the estimate

$$DV(x)f(x,d) \leq -c_1[V(x)]^a - c_2[V(x)]^b$$

is obtained. Then, from [9, Lemma 1] we know that the inequality (7) holds with $\beta(r,t) = 0$ $\forall t \geq \widetilde{T}(r)$ and that $\sup_{r \in \mathbb{R}_{\geq 0}} \widetilde{T}(r) < +\infty$. Since the estimates (24) and (25) also hold in this case, we conclude FXISS of the origin of (1). □

Proof of Corollary 2. This result follows directly from Lemma 1 in [9]. □

Proof of Theorem 5.
I Conditions **C1**, **C2** and **C4** of Theorem 2, and the implicit function theorem imply that the equation $Q(V,x) = 0$ implicitly defines a unique function $V : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}_+$ such that $Q(V(x),x) = 0$ for all $x \in \mathbb{R}^n\backslash\{0\}$.
II The function V is continuously differentiable outside the origin and $\partial_x V = -\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)}$ for $Q(V,x) = 0$, $x \neq 0$. Condition **C3** of Theorem 2 implies that the function V can be continuously prolonged at the origin (by setting $V(0) = 0$) and that V is positive definite and radially unbounded; therefore it can be bounded by

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{29}$$

where α_1, α_2 are $\mathcal{K}_\infty$ functions [1, Lemma 4.3]. Thus (5) holds.
III The derivative of V along the vector field (1) is given by

$$\begin{aligned} DV(x)f(x,d) &= \partial_x V f(x,d) \\ &= -\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d); \end{aligned}$$

and from condition $\mathbf{C5}^{\text{iss}}$ we obtain

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -\gamma(\|x\|),$$

for all $(V,x) \in \Omega$. Therefore $Q(V,x) = 0$ implicitly defines an ISS Lyapunov function for system (1). Consequently, according to Theorem 2, (1) is an ISS system.
Converse implication. If system (1) is an ISS system, then there exists an ISS Lyapunov function $\widetilde{V}: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for it (see Theorem 1). Then it is clear that the implicit ISS Lyapunov function $Q(V,x) = \frac{\widetilde{V}(x)}{V} - 1$ satisfying **C1**-$\mathbf{C5}^{\text{iss}}$ also exists. □

Proof of Theorem 6.
Sufficiency. **I** As shown before, from conditions **C1-C4** of Theorem 2, $Q(V(x),x) = 0$ implicitly defines a unique, proper, positive definite function $V(x)$ such that (29) holds and its derivative along (1) is given by

$$DV(x)f(x,d) = -\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d). \tag{30}$$

II From condition $\mathbf{C5}^{\text{ft}}$ and (30) we have that $\|x\| \geq \chi(\|d\|)$ implies that

$$DV(x)f(x,d) \leq -c[V(x)]^a$$

so that Q implicitly defines a finite-time ISS Lyapunov function. The result follows by applying Theorem 3. *Necessity.* From Theorem 3, if (1) is FTISS, there exist a finite-time ISS Lyapunov function $\widetilde{V}$. Then it is straightforward to construct an implicit finite-time ISS Lyapunov Q, *e.g.* $Q = \frac{\widetilde{V}}{V} - 1$, that satisfies conditions $\mathbf{C1}$-$\mathbf{C5}^{\text{ft}}$ hold. □

Proof of Theorem 7. By previous considerations and if condition $\mathbf{C5}^{\text{fx}}$ holds, then for $\|x\| \geq \chi(\|d\|)$

$$DV(x)f(x,d) \leq -c_1[V(x)]^a - c_2[V(x)]^b,$$

so that Q implicitly defines a fixed-time ISS Lyapunov function and from Theorem 4 we conclude that the origin of (1) is FXISS. □

Proof of Proposition 1. The proof is a direct consequence of Theorem 5 and Lemma 1. □

Proof of Proposition 2. From $\mathbf{C5}^{\text{ft}^*}$ and previous considerations we have that

$$DV(x)f(x,d) \leq \delta(\|d\|) - \kappa[V(x)]^a.$$

by adding and subtracting $\theta V(x)^a$, with $\theta \in (0, \kappa)$ we obtain

$$DV(x)f(x,d) \leq -(\kappa - \theta)[V(x)]^a - \theta[V(x)]^a + \delta(\|d\|),$$

and by taking into account (29), it becomes clear that it is always possible to find some $\chi \in \mathcal{K}_\infty$ such that

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -c[V(x)]^a,$$

take, for instance, $c = \kappa - \theta$ and $\chi = \alpha_2^{-1} \circ (\frac{1}{\theta}\delta)^{1/\alpha}$. The fixed-time case can be dealt with by following the same reasoning and repeating the arguments of the proof of Theorem 6. □

References

[1] H. K. Khalil, *Nonlinear Systems*, ser. NJ 07458. Upper Saddle River: Prentice-Hall, 1996.

[2] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351–359, 1995.

[3] S. Dashkovskiy, D. Efimov, and E. Sontag, "Input to state stability and allied system properties," *Automation and Remote Control*, vol. 72, no. 8, pp. 1579–1614, 2011.

[4] S. P. Bhat and D. S. Bernstein, "Finite time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[5] A. Bacciotti and L. Rosier, *Lyapunov Functions and Stability in Control Theory*, 2nd ed., B. Springer, Ed. Springer, 2005.

[6] H. Nakamura, Y. Yamashita, and H. Nishitani, "Smooth Lyapunov functions for homogeneous differential inclusions," in *Proceedings of the 41st SICE Annual Conference. SICE 2002.*, vol. 3. IEEE, 2002, pp. 1974–1979.

[7] M.-M. Jiang, X.-J. Xie, and K. Zhang, "Finite-time stabilization of stochastic high-order nonlinear systems with FT-SISS inverse dynamics," *IEEE Transactions on Automatic Control*, vol. 64, no. 1, pp. 313–320, 2018.

[8] E. Cruz-Zavala, J. Moreno, and L. Fridman, "Uniform robust exact differentiator," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2727–2733, 2011.

[9] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2106–2110, 2012.

[10] F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti, "Conditions for fixed-time stability and stabilization of continuous autonomous system," *Systems & Control Letters*, vol. 129, pp. 26 – 35, 2019.

[11] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, vol. 41, no. 5, pp. 823–830, 2005.

[12] J.-J. E. Slotine, W. Li *et al.*, *Applied nonlinear control*. Prentice hall Englewood Cliffs, NJ, 1991, vol. 199, no. 1.

[13] C. Edwards and S. Spurgeon, *Sliding mode control: theory and applications*. Crc Press, 1998.

[14] Y. Wu, X. Yu, and Z. Man, "Terminal sliding mode control design for uncertain dynamic systems," *Systems & Control Letters*, vol. 34, no. 5, pp. 281–287, 1998.

[15] R. Aldana-López, D. Gómez-Gutiérrez, E. Jiménez-Rodríguez, J. D. Sánchez-Torres, and M. Defoort, "Enhancing the settling time estimation of a class of fixed-time stable systems," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 12, pp. 4135–4148, 2019.

[16] R. Aldana-López, D. Gómez-Gutiérrez, M. Defoort, J. D. Sánchez-Torres, and A. J. Muñoz-Vázquez, "A class of robust consensus algorithms with predefined-time convergence under switching topologies," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 17, pp. 6179–6198, 2019.

[17] E. Jimenez-Rodriguez, A. J. M. Vázquez, J. D. Sánchez-Torres, M. Defoort, and A. G. Loukianov, "A Lyapunov-like characterization of predefined-time stability," *IEEE Transactions on Automatic Control*, 2020.

[18] Z.-Y. Sun, Y. Shao, and C.-C. Chen, "Fast finite-time stability and its application in adaptive control of high-order nonlinear system," *Automatica*, vol. 106, pp. 339–348, 2019.

[19] Z.-Y. Sun, M.-M. Yun, and T. Li, "A new approach to fast global finite-time stabilization of high-order nonlinear system," *Automatica*, vol. 81, pp. 455–463, 2017.

[20] S. Ding, K. Mei, and S. Li, "A new second-order sliding mode and its application to nonlinear constrained systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2545–2552, 2018.

[21] A. Levant, "On fixed and finite time stability in sliding mode control," in *52nd IEEE Conference on Decision and Control*. IEEE, 2013, pp. 4260–4265.

[22] D. Efimov, A. Polyakov, A. Levant, and W. Perruquetti, "Realization and discretization of asymptotically stable homogeneous systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5962–5969, 2017.

[23] A. Polyakov, D. Efimov, and B. Brogliato, "Consistent discretization of finite-time and fixed-time stable systems," *SIAM Journal on Control and Optimization*, vol. 57, no. 1, pp. 78–103, 2019.

[24] D. Efimov, A. Polyakov, and A. Aleksandrov, "Discretization of homogeneous systems using euler method with a state-dependent step," *Automatica*, vol. 109, p. 108546, 2019.

[25] Y. Hong, Z. Jiang, and G. Feng, "Finite-time input-to-state stability and applications to finite-time control design," *SIAM Journal on Control and Optimization*, vol. 48, no. 7, pp. 4395–4418, 2010.

[26] S. P. Bhat and D. S. Bernstein, "Continuous finite-time stabilization of the translational and rotational double integrator," *IEEE Trans. Automat. Control*, vol. 43, no. 5, pp. 678–682, may 1998.

[27] A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time and fixed-time stabilization: Implicit Lyapunov function approach," *Automatica*, vol. 51, pp. 332–340, 2015.

[28] H. Nakamura, N. Nakamura, and H. Nishitani, "Stabilization of homogeneous systems using implicit control lyapunov functions," *IFAC Proceedings Volumes*, vol. 40, no. 12, pp. 1120–1125, 2007.

[29] F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquettil, "On implicit finite-time and fixed-time ISS Lyapunov functions," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 706–710.

[30] E. Roxin, "On finite stability in control systems," *Rendiconti del Circolo Matematico di Palermo*, vol. 15, pp. 273–283, 1966.

[31] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Control Optim.*, vol. 34, no. 1, pp. 124–160, 1996.

[32] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals and Systems*, vol. 17, pp. 101–127, 2005.

[33] F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time and fixed-time observer design: Implicit Lyapunov function approach," *Automatica*, vol. 87, pp. 52–60, 2018.

[34] A. Polyakov and L. Fridman, "Stability notions and Lyapunov functions for sliding mode control systems," *Journal of the Franklin Institute*, vol. 351, no. 4, pp. 1831–1865, 2014.